

FRAMES FOR OPERATORS IN BANACH SPACES VIA SEMI-INNER PRODUCTS

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ABSTRACT. In this paper, we propose to define the concept of family of local atoms and then we generalize this concept to the atomic system for operator in Banach spaces by using semi-inner product. We also give a characterization of atomic systems leading to obtain new frames. In addition, a reconstruction formula is obtained. Next, some new results are established. The characterization of atomic systems allows us to state some results for sampling theory in semi-inner product reproducing kernel Banach spaces. Finally, having used frame operator in Banach spaces, new perturbation results are established.

1. INTRODUCTION

Atomic systems and K -frames, where K is a bounded linear operator on separable Hilbert space \mathcal{H} , introduced by L. Găvruta in [25] as a generalization of families of local atoms [23]. A sequence $\{f_j\}_{j \in \mathbb{N}}$ in the Hilbert space \mathcal{H} is called an *atomic system* for the bounded linear operator K on \mathcal{H} if

- (i) the series $\sum_{j \in \mathbb{N}} c_j x_j$ converges for all $c = (c_j) \in l^2 := \{\{b_j\}_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} |b_j| < \infty\}$;
- (ii) there exists $C > 0$ such that for every $f \in \mathcal{H}$ there exists $a_f = (a_j) \in l^2$ such that $\|a_f\|_{l^2} \leq \|f\|$ and $Kf = \sum_{j \in \mathbb{N}} a_j f_j$.

Also a sequence $\{f_j\}_{j \in \mathbb{N}}$ is said to be a K -frame for \mathcal{H} if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{j \in \mathbb{N}} |\langle f, f_j \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

It is proved that these two concepts are equivalent [25]. We refer to [36] for more results on these concepts. In addition, we generalized these two concepts and gave some new results [13]. Note that frames in Hilbert spaces are just a particular case of K -frames, when K is the identity operator on these Hilbert spaces. Frames in Hilbert spaces were introduced by

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Duffin and Schaffer [18] in 1952. In 1986, frames were brought to life by Daubechies *et al.* [14]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [24], sampling [19, 20], coding and communications [35], filter bank theory [4], system modeling [17], and so on.

However, the theoretical research of frames for Banach spaces is quite different from that of Hilbert spaces. Due to the lack of an inner-product, frames for Banach spaces were simply defined as a sequence of linear functionals from X^* , the dual space of X , rather than a sequence of basis-like elements in X itself. Feichtinger and Gröcheing [22] extended the notion of atomic decomposition to Banach space. Gröcheing [27] introduced a more general concept for Banach spaces called Banach frame. Aldroubi *et al.* [1] introduced p -frames and then this frame was discussed in separable Banach space in [10]. Now we are going to state frames for separable Banach spaces by Casazza *et al.* [7]. In so doing, they introduced the following definition.

Definition 1.1. A sequence space X_d is called a *BK-space*, if it is a Banach space and the coordinate functionals are continuous on X_d , i.e. the relations $x_n = \{\alpha_j^{(n)}\}$, $x = \{\alpha_j\} \in X_d$, $\lim_{n \rightarrow \infty} x_n = x$ imply $\lim_{n \rightarrow \infty} \alpha_j^{(n)} = \alpha_j$ ($j = 1, 2, 3, \dots$).

Also we add some notions about this Banach space. If the canonical vectors form a Schauder basis for X_d , then X_d is called a *CB-space* and its canonical basis is denoted by $\{e_j\}$. If X_d is reflexive and a CB-space, then X_d is called an *RCB-space*.

Frame for separable Banach space [7] is introduced as follows.

Definition 1.2. Let X be a separable Banach space and X_d be a BK-space. A countable family $\{f_j\}$ in the dual X^* is called an X_d -frame for X if

- (i) $\{f_j(f)\} \in X_d, \forall f \in X$;
- (ii) the norm $\|f\|_X$ and $\|\{f_j(f)\}\|_{X_d}$ are equivalent, i.e. there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{f_j(f)\}\|_{X_d} \leq B\|f\|_X, \quad \forall f \in X.$$

When A and B are called X_d -frame bounds. $\{f_j\}$ is called an X_d -Bessel sequence for X , if at least (i) and the upper frame condition are satisfied.

In 2011, H. Zhang and J. Zhang [38] introduced a new definition of X_d -frames by using semi-inner product. Under such a definition, an X_d -frame is exactly a sequence of elements in X itself.

The main purpose of this paper is to provide a language for the study of families of local X_d^* -atoms, X_d^* -atomic systems and X_d^* - K -frames in Banach spaces via a compatible semi-inner product, which is a natural substitute for inner products on Hilbert spaces. We obtain some new results. In particular, we characterize X_d^* - K -frames. As a consequence, we state the main result for any semi-inner product reproducing kernel Banach spaces. Our last result of this paper is to show that the Casazza–Christensen perturbation theorem [6] of Hilbert space frames somehow holds for any X_d^* - K -frames by making use X_d^* - K -frame operator.

This paper is organized as follows. In Section 2 we will give the notation used in the paper, especially the definition of semi-inner product and its properties. In Section 3 we extend notions about families of local X_d^* -atoms, X_d^* -atomic systems and X_d^* - K -frames in Banach spaces and new results are given. In section 4 by applying results from Section 3, we will present new result in reproducing kernel Banach spaces. Finally, in the last section we define X_d^* - K -frame operator and state its properties and then a perturbation of X_d^* - K -frame, the so-called Paley–Wiener perturbation [2, 8, 9], is given by use of X_d^* - K -frame operator.

2. PRELIMINARIES

We first state the following concept introduced by Lumer [29] in 1961 but its main properties discovered by Giles [26], Nath [31] and others.

A *semi-inner product* (in short s.i.p.) on a complex vector space X is a complex valued function $[f, g]$ on $X \times X$ with the following properties:

1. $[\lambda f + g, h] = \lambda[f, h] + [g, h]$ and $[f, \lambda g] = \bar{\lambda}[f, g]$, for all complex λ ,
2. $[f, f] \geq 0$, for all $f \in X$ and $[f, f] = 0$ implies $f = 0$,
3. $||[f, g]|^2 \leq [f, f][g, g]$.

It was shown in [29] that if $[\cdot, \cdot]$ is a s.i.p. on X then $\|f\| := [f, f]^{\frac{1}{2}}$ is a norm on X , and in this situation, the semi-inner product is called compatible. Conversely, if X is a normed vector space then it has a s.i.p. that induces its norm in this manner. A compatible

semi-inner product is non-additive with respect to its second variable. Namely,

$$[f, g + h] \neq [f, g] + [f, h], \quad f, g \in X,$$

in general. The concept of s.i.p. has been proved useful both theoretically and practically, and has been widely applied in the theory of functional analysis [21, 29, 32, 33], machine learning on reproducing kernel Banach spaces (in short RKBS) [37] and so on. The reader is referred to [16] for more information about semi-inner products. By properties 2 and 3 of semi-inner products, for each $f \in X$ the function that sends $g \in X$ to $[g, f]$ is a bounded linear functional on X . We shall denote this functional associated with f by f^* and call it the dual element of f . In other words, f^* lies in the dual space X^* of X . The mapping $f \rightarrow f^*$ will be called the duality mapping from X to X^* .

A Banach space X is called *strictly convex*, whenever $\|f + g\|_X = \|f\|_X + \|g\|_X$ where $f, g \neq 0$ then $f = \alpha g$ for some $\alpha > 0$. In this space the duality mapping from X to X^* is bijective [21]. In other words, for each $f^* \in X^*$ there exists a unique $g \in X$ such that $f^*(g) = [g, f]$ for all $g \in X$. Moreover, we have $\|f^*\|_{X^*} = \|f\|_X$. Also in this case, $[f^*, g^*]_* := [g, f]$, $f, g \in X$ defines a compatible semi-inner product on X^* [26].

A Banach space X is uniformly convex if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\|f + g\|_X \leq 2 - \delta$ for all $f, g \in X$ with $\|f\|_X = \|g\|_X = 1$ and $\|f - g\|_X > \epsilon$. Bear in mind that a uniformly convex Banach space is reflexive [11, page 134] but a reflexive Banach space is not necessarily uniformly convex [15]. Every uniformly convex Banach space is automatically strictly convex.

H. Zhang and J. Zhang [38] introduce X_d -frame for Banach spaces via semi-inner products as follows.

Definition 2.1. Let $[., .]$ be a compatible s.i.p. on separable strictly convex s.i.p. Banach space X and X_d be an RCB-space. Then the family $\{f_j\} \subseteq X$ is called X_d -frame for X if

- (i) $\{[f, f_j]\} \in X_d$, $\forall f \in X$;
- (ii) there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{[f, f_j]\}\|_{X_d} \leq B\|f\|_X, \quad \forall f \in X.$$

Recently, Zheng and Yang [39] have presented X_d -frame for separable uniformly convex Banach spaces via semi-inner products, when X_d is just a BK-space or CB-space.

Let $B(X, Y)$ be the bounded linear operator defined on Banach space X with values in Banach space Y . We also write $B(X)$ instead of $B(X, X)$. We say that $T \in B(X, Y)$ majorizes $Q \in B(X, Z)$ if there exists $\gamma > 0$ such that $\|Qf\| \leq \gamma \|Tf\|$ for all $f \in X$ [3]. We also recall that a closed subspace M of X is complemented if there exists a closed subspace N of X such that X can be written as the topological sum of M and N . The range of any operator T is denoted by $R(T)$. Finally, we cite some useful lemmas that will be used in the sequel as follow.

Lemma 2.2. [3] Suppose that $T \in B(X, Y)$, $U \in B(X, Z)$, and $V \in B(Z, Y)$. Then the following statements hold.

- (i) If T majorizes U and $\overline{R(T)}$ is complemented, then there exists $Q \in B(Y, Z)$ such that $U = QT$;
- (ii) Assume that T majorizes U . Then $R(U^*) \subseteq R(T^*)$;
- (iii) Assume that $R(V) \subseteq R(T)$. Then T^* majorizes U^* .

Lemma 2.3. [28, page 201] If X_d is a CB-space with the canonical unit vectors e_j , $j \in J$ then the space $X_d^\circledast := \{\{F(e_j)\}_{j \in J} : F \in X_d^*\}$ with the norm $\|\{F(e_j)\}_{j \in J}\|_{X_d^\circledast} := \|F\|_{X_d^*}$ is a BK-space isometrically isomorphic to X_d^* . Also, every continuous linear functional Ψ on X_d has the form

$$\Psi(\{c_j\}) = \sum_j c_j d_j,$$

where $\{d_j\} \in X_d^\circledast$ is uniquely determined by $d_j = \Psi(e_j)$ and

$$\|\Psi\| = \|\{\Psi(e_j)\}\|_{X_d^\circledast}.$$

When X_d^* is a CB-space then its canonical basis is denoted by $\{e_j^*\}$.

3. FAMILIES OF LOCAL X_d^* -ATOMS, X_d^* -ATOMIC SYSTEMS, AND X_d^* -K-FRAMES

Throughout this section we assume that X is a separable Banach space which is strictly convex and $[., .]$ is a compatible semi-inner product on X .

In this section we discuss families of local X_d^* -atoms, X_d^* -atomic systems, and X_d^* -K-frames in Banach spaces via semi-inner products.

First of all we give the following definition.

Definition 3.1. Let $K \in B(X)$ and X_d be a BK-space. The family $\{f_j\} \subseteq X$ is called an X_d^*-K -frame for X^* if $\{[f_j, f]\} \in X_d^*$, for all f in X and there exist $A, B > 0$ such that

$$A\|K^*f^*\|_{X^*} \leq \|\{[f_j, f]\}\|_{X_d^*} \leq B\|f^*\|_{X^*}.$$

The element A is called the lower X_d^*-K -frame bound and B is called the upper X_d^*-K -frame bound or just upper X_d^* -frame bound. If the right side of this inequality holds then we say that $\{f_j\}$ is an X_d^* -Bessel sequence for X^* . Especially, when $K = I$, the identity operator on X , then $\{f_j\}$ is called X_d^* -frame for X^* .

Now we present our definition of families of local X_d^* -atoms for Banach spaces. Actually we shall generalize the classical theory of families of local atoms for Hilbert spaces to Banach spaces via semi-inner products.

Definition 3.2. Let $\{f_j\} \subseteq X$ be an X_d^* -Bessel sequence for X^* and let X_0 be a closed subspace of X . We call $\{f_j\}$ is a *family of local X_d^* -atoms* for X_0^* if there exists a linear functional $\{\mu_j\} \subseteq X_0^*$ such that

- (i) $\{\mu_j(f)\} \in X_d$ and there exists $C > 0$ such that $\|\{\mu_j(f)\}\|_{X_d} \leq C\|f\|_X$,
 - (ii) $f = \sum_j \mu_j(f)f_j$,
- for all $f \in X_0$.

Now for the sequence $\{\mu_j\} \subseteq X_0$, by the Hahn-Banach theorem there exists $\{h_j\} \subseteq X^*$ for which $\|h_j\| = \|\mu_j\|$ and $h_j|_{X_0^*} = \mu_j$. But by the duality map from X to X^* , $h_j = g_j^*$ for some $g_j \in X$. So from the condition $\mu_j(f) = [f, g_j]$, (i) and (ii) in Definition 3.2 is equivalent to say that

- (i)' $\{g_j^*\}$ is an X_d -Bessel sequence for X_0 ,
 - (ii)' $f = \sum_j [f, g_j]f_j$,
- for all $f \in X_0$.

Proposition 3.3. Let $\{f_j\} \subseteq X$ be a family of local X_d^* -atoms for X_0^* . Then $\{f_j\}$ is an X_d^* -frame for X_0^* .

Proof. It is enough to show that $\|\{[f_j, f]\}\|_{X_d^*} \geq A\|f^*\|_{X_0^*}$, for some $A > 0$. By definition of family of local X_d^* -atoms there exists a linear functional $\{g_j^*\} \subseteq X_0^*$ such that $f = \sum_j [f, g_j]f_j$

when $\{[f, g_j]\} \in X_d$ and $\| \{[f, g_j]\} \|_{X_d} \leq C \| f^* \|_{X_0^*}$ for some $C > 0$ and for any f^* in X_0^* . So

$$\begin{aligned} \|f\|^2 &= [f, f] = \left[\sum_j [f, g_j] f_j, f \right] = \sum_j [f, g_j] [f_j, f] \\ &\leq \| \{[f, g_j]\} \|_{X_d} \| \{[f_j, f]\} \|_{X_d^*} \leq C \| f^* \| \| \{[f_j, f]\} \|_{X_d^*}. \end{aligned} \quad (3.1)$$

This implies that $\| \{[f_j, f]\} \|_{X_d^*} \geq \frac{1}{C} \| f^* \|_{X_0^*}$. Thus $\{f_j\}$ is an X_d^* -frame for X_0^* with lower X_d^* -frame bound $\frac{1}{C}$. \square

To generalize the concept of a family of local X_d^* -atoms, we state the following lemma.

Lemma 3.4. *Let X_d be a CB-space and $\{f_j\} \subseteq X$. If $\sum_j b_j f_j$ converges in X , for all $b = \{b_j\}$ in X_d and $\| \sum_j b_j f_j \|_X \leq B \|b\|_{X_d}$, then $\{f_j\}$ is an X_d^* -Bessel sequence for X^* with bound B .*

Proof. Since $\sum_j b_j f_j$ converges in X for every $b = \{b_j\}$ in X_d , we can define the operator $U : X_d \rightarrow X$ by $U(\{b_j\}) = \sum_j b_j f_j$. Then we have $\|U\| \leq B$, $U(e_j) = f_j$ and its dual operator is a bounded operator which is defined by $U^* : X^* \rightarrow X_d^*$ satisfies:

$$U^*(f^*)(e_j) = f^*(U(e_j)) = f^*(f_j) = [f_j, f].$$

So by Lemma 2.3, $\{U^*(f^*)(e_j)\} = \{[f_j, f]\}$ is identified with $U^* f^*$. Therefore, $\| \{[f_j, f]\} \|_{X_d^*} = \| U^* f^* \| \leq \|U\| \|f\| \leq B \|f\|$. It means, $\{f_j\}$ is an X_d^* -Bessel sequence for X^* with bound B . \square

The generalization of a family of local X_d^* -atoms is given below.

Definition 3.5. Let $\{f_j\} \subseteq X$ and $K \in B(X)$. $\{f_j\}$ is called an X_d^* -atomic system for X^* with respect to K if

- (i) $\sum_j b_j f_j$ converges in X for all $b = \{b_j\}$ in X_d and there exists $B > 0$ such that $\| \sum_j b_j f_j \|_X \leq B \|b\|_{X_d}$;
- (ii) There exists $C > 0$ such that for every $f \in X$ there exists $a_f = \{a_j\} \in X_d$ such that $\|a_f\| \leq C \|f\|$ and $Kf = \sum_j a_j f_j$.

Indeed, every family of local X_d^* -atoms is an X_d^* -atomic system. Note that the part (i) says that $\{f_j\}$ is an X_d^* -Bessel sequence for X^* by Lemma 3.4. One of the most important results of this paper is given below. Actually, this is a relation between X_d^* -atomic systems and X_d^* - K -frames. Besides, a new reconstruction is obtained.

Theorem 3.6. Let X_d be a CB -space and $\{f_j\}$ be an X_d^* -Bessel sequence for X^* . Suppose that $T : X^* \rightarrow X_d^*$ is given by $T(f^*) = \{[f_j, f]\}$ and $\overline{R(T)}$ is complemented in X_d^* . Then the following statements are equivalent:

- (i) $\{f_j\}$ is an X_d^* -atomic system for X^* with respect to K ;
- (ii) $\{f_j\}$ is an X_d^* - K -frame for X^* ;
- (iii) $\{f_j\}$ is an X_d^* -Bessel sequence for X^* and there exists an X_d -Bessel sequence $\{g_j^*\}$ for X such that for any $f^* \in X^*$, we have

$$K^*f^* = \sum_j [f_j, f]g_j^*.$$

Proof. (i) \Rightarrow (ii). For every $f^* \in X^*$, we have

$$\begin{aligned} \|K^*f^*\| &= \sup_{g \in X, \|g\|=1} |K^*f^*(g)| \\ &= \sup_{g \in X, \|g\|=1} |f^*(Kg)| = \sup_{g \in X, \|g\|=1} |[Kg, f]|, \end{aligned}$$

by definition of an X_d^* -atomic system there exists $\{b_j\} \in X_d$ such that $\|\{g_j\}\| \leq C\|g\|$, for some $C > 0$, $Kg = \sum_j b_j f_j$, so

$$\begin{aligned} \|K^*f^*\| &= \sup_{g \in X, \|g\|=1} \left| \sum_j b_j [f_j, f] \right| \\ &\leq \sup_{g \in X, \|g\|=1} \|\{b_j\}\|_{X_d} \|\{[f_j, f]\}\|_{X_d^*} \leq C \|\{[f_j, f]\}\|_{X_d^*}. \end{aligned}$$

Therefore the lower X_d^* - K -frame bound of $\{f_j\}$ is $\frac{1}{C}$, i.e. $\frac{1}{C}\|K^*f^*\| \leq \|\{[f_j, f]\}\|_{X_d^*}$.

(ii) \Rightarrow (iii). Since T majorizes K^* and $\overline{R(T)}$ is complemented in X_d^* , by Lemma 2.2 there exists a bounded operator $Q : X_d^* \rightarrow X^*$ such that $K^* = QT$. So

$$K^*(f^*) = QT(f^*) = Q\left(\sum_j [f_j, f]e_j^*\right) = \sum_j [f_j, f]Qe_j^*,$$

for $Qe_j^* \in X^*$ there exists a unique $g_j \in X$ such that $g_j^* = Qe_j^*$. So by Lemma 2.3 we have,

$$\|\{[g_j^*, g^*]\}\|_{X_d} = \|\{g(g_j^*)\}\| = \|\{g(Q(e_j^*))\}\| = \|\{Q^*(g)(e_j^*)\}\| = \|Q^*(g)\| \leq \|Q\|\|g\|_X,$$

it means, $\{g_j^*\}$ is an X_d -Bessel sequence for X .

(iii) \Rightarrow (i). Suppose that $\{f_j\}$ is an X_d^* -Bessel sequence for X^* with bound B and $b = \{b_j\} \in X_d$. We estimate for positive integers $m > n$ that

$$\begin{aligned} \left\| \sum_{j \in J_m \setminus J_n} b_j f_j \right\| &= \sup_{f^* \in X^*, \|f^*\| \leq 1} \left| f^* \left(\sum_{j \in J_m \setminus J_n} b_j f_j \right) \right| \\ &= \sup_{f^* \in X^*, \|f^*\| \leq 1} \left| \sum_{j \in J_m \setminus J_n} b_j [f_j, f] \right| \\ &\leq \left\| \sum_{j \in J_m \setminus J_n} b_j e_j \right\|_{X_d} \sup_{f^* \in X^*, \|f^*\| \leq 1} \|\{[f_j, f]\}\|_{X_d^*} \\ &\leq B \left\| \sum_{j \in J_m \setminus J_n} b_j e_j \right\|_{X_d}, \end{aligned} \tag{3.2}$$

as e_j form a Schauder basis for X_d , $\|\sum_{j \in J_m \setminus J_n} b_j e_j\|_{X_d}$ goes to zero as m, n tend to infinity. As a result $\sum_{j \in J} b_j f_j$ converges in X .

Using the same technique as that engaged in (3.2), we obtain

$$\left\| \sum_j b_j f_j \right\|_X \leq B \|b\|_{X_d}.$$

Now assume that $g \in X$ and $f^* \in X^*$. Then we have

$$\begin{aligned} (Kg)(f^*) &= g(K^* f^*) = [K^* f^*, g^*]_* = [g, (K^* f^*)^*] \\ &= (K^* f^*)(g) = (\sum_j [f_j, f] g_j^*)(g) = \sum_j [f_j, f] [g, g_j] \\ &= \sum_j [g, g_j] [f^*, f_j^*]_* = \sum_j [g, g_j] f_j(f^*). \end{aligned}$$

Therefore $Kf = \sum_j a_j f_j$, when $a_f = \{a_j\} = \{[f, g_j]\}$. Note that, since $\{g_j^*\}$ is an X_d -Bessel sequence for X , there exists $C > 0$ such that $\|a_f\| \leq C \|f\|$. \square

Now we present an example in order to describe our work.

Example 3.7. Consider the space $X := \ell^{\frac{3}{2}}(\mathbb{N}_3)$ with the semi-inner product

$$[g, h] := \|g\|^{\frac{1}{2}} \sum_{j=1}^3 g_j \overline{h_j} |h_j|^{-\frac{1}{2}}, \quad g, h \in X,$$

and the following sequence in X :

$$f_1 = e_1, f_2 = e_2, f_3 = 0.$$

We can easily show that for BK-space $X_d^* := \ell^3(\mathbb{N})$, $\{f_1^* = e_1, f_2^* = e_2, f_3^* = 0\}$ is not an X_d^* -frame for X^* . Now we define a bounded linear operator $K^* : X^* \rightarrow X^*$ as follows:

$$K^*e_1 = e_1, K^*e_2 = e_2, K^*e_3 = 0.$$

We show that $\{f_1^*, f_2^*, f_3^*\}$ is an X_d^* - K -frame for X^* . In so doing, we have

$$\|K^*f^*\|_{\ell^3(\mathbb{N})} = \|c_1e_1 + c_2e_2\|_{\ell^3(\mathbb{N})} = (|c_1|^3 + |c_2|^3)^{\frac{1}{3}} \leq \|\{[f_1, f], [f_2, f], [f_3, f]\}\|_{\ell^3(\mathbb{N})},$$

where $f^* = c_1e_1 + c_2e_2 + c_3e_3$, for some $c_1, c_2, c_3 \in \mathbb{C}$.

A characterization of an X_d^* - K -frame is given below.

Theorem 3.8. *Let X_d be a CB-space. Then $\{f_j\}$ is an X_d^* - K -frame if and only if there exists a bounded linear operator $U : X_d \rightarrow X$ such that $Ue_j = f_j$ and $R(K) \subseteq R(U)$.*

Proof. Since $\{f_j\}$ is an X_d^* - K -frame, we can define $U : X_d \rightarrow X$ by $U(c) = \sum_j c_j f_j$, $c = \{c_j\} \in X_d$. Therefore $Ue_j = f_j$ and U is bounded, i.e. $\|U\| \leq B$, where B is the upper X_d^* - K -frame bound of $\{f_j\}$. By the similar way of Lemma 3.4 $\{U^*(f^*)(e_j)\} = \{[f_j, f]\}$ is identified with U^*f^* for every $f^* \in X^*$. Therefore

$$\|\{[f_j, f]\}\|_{X_d^*} = \|U^*f^*\| \leq B\|f^*\|_{X^*}.$$

Now by $\|K^*f^*\| \leq \|\{[f_j, f]\}\|_{X_d^*} = \|U^*f^*\|$ and Lemma 2.2 we have $R(K) \subseteq R(U)$.

Conversely, by the similar way $\{U^*(f^*)(e_j)\} = \{[f_j, f]\}$ is identified with U^*f^* , $f^* \in X^*$. Therefore

$$\|\{[f_j, f]\}\|_{X_d^*} = \|U^*f^*\| \leq \|U\|\|f^*\|_{X^*}.$$

Since $R(K) \subseteq R(U)$ then by Lemma 2.2 there exists $A > 0$ such that $A\|K^*f^*\| \leq \|U^*f^*\| = \|U^*(f^*)(e_j)\| = \|\{[f_j, f]\}\|$. It means, $\{f_j\}$ is an X_d^* - K -frame. \square

In the following part some results are given.

Proposition 3.9. *Suppose that $\{f_j\}$ is an X_d^* -frame for X^* and $Q \in B(X)$. Then $\{Qf_j\}$ is an X_d^* -frame for X^* if and only if Q^* is bounded below.*

Proof. Let $f \in X$ then we have

$$[Qf_j, f] = f^*(Qf_j) = Q^*f^*(f_j) = [f_j, (Q^*f^*)^*]. \quad (3.3)$$

Let $\{f_j\}$ be an X_d^* -frame for X^* with upper X_d^* -frame bound B and $\{Qf_j\}$ be an X_d^* -frame for X^* with lower X_d^* -frame bound C . By (3.3) we have

$$C\|f^*\| \leq \|\{[Qf_j, f]\}\| = \|\{[f_j, (Q^*f^*)^*]\}\| \leq B\|Q^*f^*\|,$$

therefore $\|Q^*f^*\| \geq \frac{C}{B}\|f^*\|$, i.e. Q^* is a bounded below operator.

Now let $\{f_j\}$ be an X_d^* -frame for X^* with X_d^* -frame bounds A and B . Then $\|\{[f_j, (Q^*f^*)^*]\}\| \leq B\|Q^*f^*\| \leq B\|Q\|\|f^*\|$. On the other hand $\|\{[f_j, (Q^*f^*)^*]\}\| \geq A\|Q^*f^*\|$, since Q^* is bounded below, there exists $\gamma > 0$ such that $\|Q^*f^*\| \geq \gamma\|f^*\|$. Therefore $\|\{[f_j, (Q^*f^*)^*]\}\| \geq A\gamma\|f^*\|$. Hence by (3.3) $\{Qf_j\}$ is an X_d^* -frame for X^* . \square

The following two propositions are proved by the similar way of Proposition 3.9.

Proposition 3.10. *Let $K \in B(X)$ and $\{f_j\}$ be an X_d^* -frame for X^* with X_d^* -frame bounds A and B , respectively. Then $\{Kf_j\}$ is an X_d^* - K -frame for X^* with X_d^* - K -frame bounds A and $B\|K\|$, respectively.*

Proposition 3.11. *Let $\{f_j\}$ be an X_d^* - K -frame for X^* . Then $\{f_j\}$ is an X_d^* -frame for X^* if K is a bounded below operator.*

4. SAMPLING IN A S.I.P. RKBS

The main result of previous section in any s.i.p. RKBSs is given in this section. First, we state some notations needed for our next result.

We mention that X is *uniformly Fréchet differentiable* if for all $f, g \in X$

$$\lim_{\lambda \in \mathbb{R}, \lambda \rightarrow 0} \frac{\|f + \lambda g\|_X - \|f\|_X}{\lambda} \quad (4.1)$$

exists and the limit is approached uniformly for all f, g in the unit ball of X . If X is uniformly Fréchet differentiable, then it has a unique compatible semi-inner product [27] and see also [37]. The differentiability (4.1) of the norm is useful to derive characterization equations for the minimizer of regularized learning schemes in Banach spaces. For simplicity, we call a Banach space uniform if it is both uniformly convex and uniformly Fréchet differentiable. Notice that its dual X^* is also uniform [12]. In this section, X is called a s.i.p. RKBS on Ω if it is a uniform Banach space of functions on Ω where point evaluations are always continuous linear functionals. Also its s.i.p. reproducing kernel is denoted by k . For the theory of RKBSs see for instance [37] and references therein. Most important of all, by the

arguments in the proof of Theorem 9 in [37], there exists a unique function $G : X \times X \rightarrow \mathbb{C}$ such that $G(t, \cdot) \in X$ for all $t \in \Omega$ and $f(t) = [f, G(t, \cdot)]$, for all $t \in \Omega$ and $f \in X$. By virtue of the above equation, G is called the s.i.p. reproducing kernel of X . Moreover, there holds the relationship $k(\cdot, t) = (G(t, \cdot))^*$, $t \in \Omega$ and $f^*(t) = [k(t, \cdot), f]$ for all $f \in X$ and $t \in \Omega$. Set $K_{\mathcal{Z}} := \{G(\cdot, t_j)^*\} = \{k(t_j, \cdot)\}$. The sampling operator $\mathcal{I}_{\mathcal{Z}} : X^* \rightarrow X_d^*$ is defined by $\mathcal{I}_{\mathcal{Z}}(f^*) = \{f^*(t_j)\}$, i.e. $\mathcal{I}_{\mathcal{Z}}(f^*) = \{[k(t_j, \cdot), f]\}, t_j \in \Omega$. For more details one can see page 12 to 14 [38]. Now the result of main Theorem in any s.i.p. RKBS is given below.

Theorem 4.1. *With the notations mentioned above, let X_d be a CB-space and X be a s.i.p. RKBS on Ω with the s.i.p. reproducing kernel k and $K_{\mathcal{Z}}$ be an X_d^* -Bessel sequence for X^* and $\overline{\mathcal{I}_{\mathcal{Z}}(X^*)}$ is complemented in X_d^* . Then the following statements are equivalent:*

- (i) $K_{\mathcal{Z}}$ is an X_d^* -atomic system for X^* with respect to K ;
- (ii) $K_{\mathcal{Z}}$ is an X_d^* - K -frame for X^* , i.e. there exist $A, B > 0$ such that

$$A\|K^*f^*\|_{X^*} \leq \|\mathcal{I}_{\mathcal{Z}}(f^*)\|_{X_d^*} \leq B\|f^*\|_{X^*}, \quad \text{for all } f^* \in X^*;$$

- (iii) $K_{\mathcal{Z}}$ is an X_d^* -Bessel sequence for X^* and there exists an X_d -Bessel sequence $\{g_j^*\}$ for X such that for any $f \in X$ we have

$$K^*f^* = \sum_j f^*(t_j)g_j^* = \sum_j [k(t_j, \cdot), f]g_j^*.$$

A corollary of the previous Theorem is given below.

Corollary 4.2. *With the notations mentioned above, let X_d be a CB-space and X be a s.i.p. RKBS on Ω with the s.i.p. reproducing kernel k and $K_{\mathcal{Z}}$ be an X_d^* -Bessel sequence for X^* and $\overline{\mathcal{I}_{\mathcal{Z}}(X^*)}$ is complemented in X_d^* . Then the following statements are equivalent:*

- (i) $K_{\mathcal{Z}}$ is an X_d^* -frame for X^* ;
- (ii) $K_{\mathcal{Z}}$ is an X_d^* -Bessel sequence for X^* and there exists an X_d -Bessel sequence $\{g_j^*\}$ for X such that for any $f \in X$, we have

$$f^* = \sum_j f^*(t_j)g_j^* = \sum_j [k(t_j, \cdot), f]g_j^*.$$

5. PERTURBATIONS OF X_d^* - K -FRAMES BY USING X_d^* - K -FRAME OPERATORS

Several authors have generalized the Paley–Wiener perturbation theorem to the perturbation of frames in Hilbert spaces. The most general result of these was obtained by Casazza

and Christensen [6, Theorem 2.1]. We show that the Casazza–Christensen perturbation theorem of Hilbert space frames somehow holds for X_d^* - K -frames. In order to state and prove this theorem, we have to define X_d^* - K -frame operator for X^* . The first thing which we will employ is a known result about pseudo-inverse of any bounded linear operator [5, 30].

Let X and Y be Banach spaces and $Q \in B(X, Y)$ be such that the range $R(Q)$ of Q is closed in Y . Assume that X is the topological sum of the null space $N(Q)$ of Q and $N(Q)^c$, and Y is the topological sum of $R(Q)$ and $R(Q)^c$, where $N(Q)^c$ and $R(Q)^c$ are closed subspaces of X and Y , respectively. Note that Q is one-to-one from $N(Q)^c$ onto $R(Q)$. Let P_X be the projection of X onto $N(Q)$ along $N(Q)^c$, and let P_Y be the projection of Y onto $R(Q)$ along $R(Q)^c$. The bounded linear operator $Q^\dagger : Y \rightarrow X$ defined by $Q^\dagger Qf = f$ for $f \in N(Q)^c$ and $Q^\dagger g = 0$ for $g \in R(Q)^c$ is called the pseudo-inverse of Q (with respect to P_X, P_Y). In particular, for any $g \in R(Q)$, $QQ^\dagger g = g$. If there exists a pseudo-inverse Q^\dagger of Q such that $QQ^\dagger f = f$, for any f in $R(Q)$, namely $QQ^\dagger|_{R(Q)} = I|_{R(Q)}$, then $(Q^\dagger|_{R(Q^*)})^*Q^* = I|_{R(Q^*)}$. So we have $\|f^*\| = \|(Q^\dagger|_{R(Q^*)})^*Q^*f^*\| \leq \|Q^\dagger\|\|Q^*f^*\|$ for every $f^* \in R(Q^*)$. In the rest of this section when we use K^\dagger , in any X_d^* - K -frame, we mean, under these conditions.

Now, we are going to define X_d^* - K -frame operator (see also [34]). Let X be a separable Banach space and X_d be a BK-space. In order to compose the operator $T : X^* \rightarrow X_d^*$ defined by $T(f^*) = \{[f_j, f]\}$ and the operator $U : X_d \rightarrow X$ defined by $U(\{c_j\}) = \sum_j c_j f_j$, we use the duality mapping $\Phi_{X_d^*} : X_d^* \rightarrow X_d^{**}$, $\Phi_{X_d^*}(c^*) = \{c^{**} \in X_d^{**} : c^{**}(c^*) = \|c^*\|^2 = \|c^{**}\|^2\}$, in the case when it is single-valued.

For a given $\{f_j\}$, as an X_d^* - K -frame for X^* , the operators T and U , defined above, call the *analysis* and *synthesis* operator for $\{f_j\}$. In the sequel we use these operators as they are defined above.

The duality mapping $\Phi_{X_d^*}$ needs being single-valued. In so doing, X_d^* or just X_d has to be uniformly convex BK-space because the bi-dual element of c , in any uniformly convex Banach space X_d , equals to c , i.e. $c^{**} = c$. Besides, if X_d^{**} is strictly convex then $\Phi_{X_d^*}$ is single-valued [16]. Note that a uniformly convex Banach space is automatically strictly convex and reflexive. In addition, a dual space of any Banach space is also uniformly convex Banach space. Therefore, $\Phi_{X_d^*}$ is a single-valued map to X_d . In the sequel, if there is no risk of confusion, we will omit the index and write Φ instead of $\Phi_{X_d^*}$.

Definition 5.1. Let X be a strictly convex separable Banach spaces, X_d be a uniformly convex BK-space, and $\{f_j\}$ be an X_d^* - K -frame for X^* . We define the X_d^* - K -frame operator for $\{f_j\}$ via

$$S := U\Phi_{X_d^*}T.$$

Note that S is the bounded operator from X^* to X .

Lemma 5.2. Let X_d be a CB-space and $\{f_j\}$ be an X_d^* -Bessel sequence for X^* , then $U^* = T$.

Proof. Let $f^* \in X^*$, then

$$U^*(f^*)(e_j) = f^*(Ue_j) = f^*(f_j) = [f_j, f] = T(f^*)(e_j).$$

For $c = \{c_j\} \in X_d$ we have

$$\begin{aligned} U^*(f^*)(c) &= U^*(f^*)(\sum_j c_j e_j) = \sum_j c_j U^*(f^*)(e_j) \\ &= \sum_j c_j T(f^*)(e_j) = T(f^*)(\sum_j c_j e_j) = T(f^*)(c), \end{aligned}$$

therefore, $U^* = T$. \square

Under the assumption in Lemma 5.2 S , the X_d^* - K -frame for $\{f_j\}$, can be written $U\Phi U^*$ instead of $U\Phi T$. In the sequel we use S as the X_d^* - K -frame operator for $\{f_j\}$.

The following proposition of X_d^* - K -frame operators need for our next study.

Proposition 5.3. Let X be a strictly convex separable Banach space, X_d be a uniformly convex BK-space, and $\{f_j\}$ be an X_d^* - K -frame for X^* with X_d^* - K -frame bounds A and B , respectively. Then the following statement holds:

$$(i) A^2 \|K^* f^*\|^2 \leq [Sf^*, f] \leq B^2 \|f^*\|^2, \text{ for all } f^* \in X^*.$$

Moreover, suppose that K has pseudo-inverse K^\dagger . For any $f^* \in R(K^*)$, the following statements hold:

$$(ii) A^2 \|K^\dagger\|^{-2} \|f^*\| \leq \|Sf^*\| \leq B^2 \|f^*\|;$$

$$(iii) \|Tf^*\| \leq A^{-1} \|K^\dagger\| \|Sf^*\|.$$

Proof. (i) Since $\{f_j\}$ is an X_d^* - K -frame for X^* , it is enough to show that $[Sf^*, f] = \|\{[f_j, f]\}\|_{X_d^*}^2$. For $f^* \in X^*$ we have

$$\begin{aligned}[Sf^*, f] &= [U\Phi Tf^*, f] = f^*(U\Phi Tf^*) = U^*f^*(\Phi Tf^*) \\ &= (\{[f_j, f]\})(\{[f_j, f]\})^* = [(\{[f_j, f]\})^*, (\{[f_j, f]\})^*]_{X_d} \\ &= [\{[f_j, f]\}, \{[f_j, f]\}]_{X_d^*} = \|\{[f_j, f]\}\|_{X_d^*}^2.\end{aligned}$$

(ii) From part (i) and $\|f^*\| = \|(K^\dagger|_{R(K^*)})^*K^*f^*\| \leq \|K^\dagger\| \|K^*f^*\|$ for every $f^* \in R(K^*)$, we obtain

$$[Sf^*, f] \geq A^2 \|K^*f^*\|^2 \geq A^2 \|K^\dagger\|^{-2} \|f^*\|^2,$$

hence

$$\|Sf^*\| \geq \frac{[Sf^*, f]}{\|f\|} \geq \frac{A^2 \|K^\dagger\|^{-2} \|f^*\|^2}{\|f\|} = A^2 \|K^\dagger\|^{-2} \|f^*\|. \quad (5.1)$$

On the other hand

$$\begin{aligned}\|Sf^*\| &= \sup_{g^* \in X^*, \|g^*\|=1} g^*(Sf^*) = \sup_{g \in X, \|g\|=1} [Sf^*, g] \\ &= \sup_{g \in X, \|g\|=1} \left[\sum_j d_j f_j, g \right] = \sup_{g \in X, \|g\|=1} \sum_j d_j [f_j, g] \\ &\leq \sup_{g \in X, \|g\|=1} \|\{d_j\}\| \|\{[f_j, g]\}\| \leq \sup_{g \in X, \|g\|=1} B \|g\| \|\{d_j\}\| \\ &\leq B \|\{[f_j, f]\}\| \leq B^2 \|f^*\|,\end{aligned} \quad (5.2)$$

where $\{d_j\} := \{[f_j, f]\}^*$, for all $j \in J$. Therefore, we get part (ii) by applying (5.1) and (5.2).

(iii) By applying part (i) and (ii) we have

$$\begin{aligned}\|Tf^*\|^2 &= \|\{[f_j, f]\}\|^2 = [Sf^*, f] \\ &\leq \|Sf^*\| \|f\| \leq A^{-2} \|K^\dagger\|^2 \|Sf^*\|^2\end{aligned}$$

This implies part (iii). □

We are now ready to state and prove our theorem about perturbation of X_d^* - K -frame. Actually, it is the generalization of [6, Theorem 2.1], [36, Theorem 3.13], and [39, Proposition 4.3].

Theorem 5.4. Assume that X is a strictly convex separable Banach space, X_d is a uniformly convex Banach space, and $\{f_j\}$ is an X_d^* - K -frame for X^* with X_d^* -frame bounds A and B , respectively. Suppose that $\{g_j\}$ is a sequence in X . If there exist $\alpha, \beta, \gamma \geq 0$ such that for any finite sequence $\{c_j\} \in X_d$,

$$\left\| \sum_j c_j(g_j - f_j) \right\|_X \leq \alpha \left\| \sum_j c_j f_j \right\|_X + \beta \left\| \sum_j c_j g_j \right\|_X + \gamma \left\| \{c_j\} \right\|_{X_d}, \quad (5.3)$$

is fulfilled with $\max\{\beta, \alpha + \gamma A^{-1} \|K^\dagger\| \|\Phi\|\} < 1$. Then $\{g_j\}$ is also an X_d^* -PK-frame for X^* , where P is the orthogonal projection operator from X to $Q(R(K^*))$, $Q := V\Phi U^*$, U , V are synthesis operators for $\{f_j\}$ and $\{g_j\}$, respectively.

Proof. Since $\{f_j\}$ is an X_d^* - K -frame for X^* , we can define operators $U : X_d \rightarrow X$ by

$$U(\{c_j\}) = \sum_j c_j f_j,$$

Furthermore, $\|U\| \leq B$. Suppose that condition (5.3) holds for any finite sequence $\{c_j\}$. Then for each $c = \{c_j\} \in X_d$ we have that

$$\left\| \sum_j c_j g_j \right\| \leq \frac{1+\alpha}{1-\beta} \left\| \sum_j c_j f_j \right\| + \frac{\gamma}{1-\beta} \left\| \{c_j\} \right\|.$$

So

$$\left\| \sum_{j \in J_m \setminus J_n} c_j g_j \right\| \leq \frac{(1+\alpha)B + \gamma}{1-\beta} \left\| \sum_{j \in J_m \setminus J_n} c_j e_j \right\|.$$

When $m > n$ tend to ∞ , $\sum_{j \in J_m \setminus J_n} c_j e_j$ tends to zero. Thus, the series $\sum_j c_j g_j$ converges for any $\{c_j\} \in X_d$ and

$$\left\| \sum_j c_j g_j \right\| \leq \frac{(1+\alpha)B + \gamma}{1-\beta} \left\| \sum_j c_j e_j \right\|.$$

By Lemma 3.4, $\{g_j\}$ is an X_d^* -Bessel sequence for X^* with bound $\frac{(1+\alpha)B + \gamma}{1-\beta}$.

Now we show that $\{g_j\}$ has a lower X_d^* - K -frame bound. The condition (5.3) turns to be

$$\|Uc - Vc\| \leq \alpha \|Uc\| + \beta \|Vc\| + \gamma \|c\|, \quad c = \{c_j\} \in X_d.$$

For $c = \Phi_{X_d^*} U^* f^* \in X_d$ we have

$$\|U\Phi U^* f^* - V\Phi U^* f^*\| = \|Sf^* - V\Phi U^* f^*\| \quad (5.4)$$

$$\leq \alpha \|Sf^*\| + \beta \|V\Phi U^* f^*\| + \gamma \|\Phi U^* f^*\|, \quad (5.5)$$

Here we use Lemma 5.2, i.e. $S = U\Phi_{X_d^*}U^*$. By part (iii) of Proposition 5.3 and (5.4) we have

$$\|Sf^* - V\Phi U^* f^*\| \leq (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|)\|Sf^*\| + \beta\|V\Phi U^* f^*\|. \quad (5.6)$$

By triangular inequality, it follows from (5.6) that

$$\begin{aligned} \frac{1 - (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|)}{1 + \beta}\|Sf^*\| &\leq \|V\Phi U^* f^*\| \\ &\leq \frac{1 + \alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|}{1 - \beta}\|Sf^*\|. \end{aligned} \quad (5.7)$$

By Combination part (ii) of Proposition 5.3 and (5.7), for any $f^* \in R(K^*)$, we have

$$\begin{aligned} &\frac{(1 - (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|))A^2\|K^\dagger\|^{-2}}{1 + \beta}\|f^*\| \leq \|V\Phi U^* f^*\| \\ &\leq \frac{(1 + \alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|)B^2}{1 - \beta}\|f^*\|. \end{aligned} \quad (5.8)$$

Next we show that $R(Q := V\Phi U^*)$ is closed. Suppose that $\{f_n\}_{n=1}^\infty \subseteq R(Q)$ and $f_n \rightarrow f$ as $n \rightarrow \infty$. Then we can find $g_n \in R(K^*)$ such that $Q(g_n) = f_n$. It follows from inequality (5.8) that $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence. Suppose that $g_n \rightarrow g$ as $n \rightarrow \infty$. Therefore $f_n = Qg_n \rightarrow Qg = f$ as $n \rightarrow \infty$. Which implies that $R(Q)$ is closed. From (5.8), $Q : R(K^*) \rightarrow R(Q)$ is invertible. By (5.8) we obtain that for any $f \in Q(R(K^*))$,

$$\|Q^{-1}(f)\| \leq \frac{1 + \beta}{(1 - (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|))A^2\|K^\dagger\|^{-2}}\|f\|. \quad (5.9)$$

On the other hand, for any $f \in X$, we also have

$$Pf = QQ^{-1}Pf = V(\Phi U^* Q^{-1} Pf) = \sum_j (\Phi U^* Q^{-1} Pf)_j g_j$$

Hence for arbitrary $f^* \in X^*$, we get

$$\begin{aligned}
\|K^*P^*f^*\| &= \sup_{g \in X, \|g\|=1} \|(K^*P^*f^*)(g)\| = \sup_{g \in X, \|g\|=1} \|f^*(PKg)\| \\
&= \sup_{g \in X, \|g\|=1} |[PKg, f]| = \sup_{g \in X, \|g\|=1} |[\sum_j (\Phi U^* Q^{-1} PKg)_j g_j, f]| \\
&= \sup_{g \in X, \|g\|=1} |\sum_j (\Phi U^* Q^{-1} PKg)_j [g_j, f]| \\
&\leq \sup_{g \in X, \|g\|=1} \|\Phi U^* Q^{-1} PKg\|_{X_d} \|[g_j, f]\|_{X_d^*} \\
&\leq \sup_{g \in X, \|g\|=1} \|\Phi\| \|U^*\| \|Q^{-1} PKg\| \|[g_j, f]\|_{X_d^*} \\
&\leq \sup_{g \in X, \|g\|=1} \frac{B\|\Phi\|(1+\beta)}{1 - (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|)A^2\|K^\dagger\|^{-2}} \|K\| \|g\| \|[g_j, f]\|_{X_d^*},
\end{aligned}$$

so we obtain the lower X_d^* - K -frame bound condition,

$$\frac{1 - (\alpha + \gamma A^{-1}\|K^\dagger\|\|\Phi\|)A^2\|K^\dagger\|^{-2}}{B\|\Phi\|(1+\beta)\|K\|} \|(PK)^*f^*\| \leq \|[g_j, f]\|.$$

Therefore, $\{g_j\}$ is the X_d^* - PK -frame for X^* . \square

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